Drinfeld's best hope & abelianization

C smooth proj curve / $k = \mathbb{C}$ $\qquad G = Gl_n(\mathbb{C})$

Recall:

$$\left\{ \begin{array}{l} \text{irred. rank } n \\ \text{local systems} \\ E = (\mathcal{E}, \nabla) \text{ on } C \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Hecke eigen-}\mathcal{D}\text{-modules} \\ Aut_E \text{ on } Bun_n, \text{ i.e.} \\ H^i(Aut_E) \simeq \Lambda^i(E) \boxtimes Aut_E \end{array} \right\}$$

$$\cap \qquad\qquad\qquad\qquad$$

$$Loc_n(\mathbb{C}) \qquad\qquad \text{(Drinfeld, FGV)}$$

Categorify:

LHS $\leftrightarrow$ skyscraper sheaves on pt^s of moduli stack Loc_n

NB: These are eigensheaves for Wilson operators $W^i := \Lambda^i(E^{univ}) \boxtimes (-)$

Drinfeld's best hope: $\exists$ natural equivalence $\quad \left(\begin{array}{l} E^{univ} \text{ universal sheaf} \\ \text{on } C \times Loc_n \end{array} \right)$

$$\begin{array}{ccc} D^b_{coh}(\mathcal{O}_{Loc_n}) & \xrightarrow{\sim} & D^b_{coh}(\mathcal{D}_{\mathcal{B}un_n}) \\ \circlearrowleft & & \circlearrowleft \\ W^i & & H^i \end{array} \qquad \mathcal{B}un_n := Bun_n /\!\!/ \mathbb{G}_m$$

Rem. (a) $\qquad\qquad$ disconnected

$$Bun_n = \coprod_{d \in \mathbb{Z}} Bun_n^d$$

but Loc_n irreducible (flatness!)

But $Z(G) = \mathbb{G}_m$ acts on every point of $\mathcal{L}oc_n$

$$\Rightarrow D(\mathcal{O}_{\mathcal{L}oc_n}) = \bigoplus_{d \in \mathrm{Hom}(Z(G), \mathbb{G}_m) = \mathbb{Z}} D(\mathcal{O}_{\mathcal{L}oc_n}, d)$$

$\uparrow$ d-twisted sheaves
ie $z \in Z(G)$ acts
on stalks by $z^d \cdot \mathrm{id}$.

H^i shifts degree by i

W^i shifts the twist by $-i$

(since E^{univ} is 1-twisted)

$$\Rightarrow \text{Expect} \quad D(\mathcal{O}_{\mathcal{L}oc_n}, -d) \xrightarrow{\sim} D(\mathcal{D}_{\mathcal{B}un_n^d})$$

where $\mathcal{B}un_n := \mathcal{B}un_n /\!\!/ Z(G)$

Slogan: Disconnectedness $\longleftrightarrow$ Gerbiness via duality!

& we need stacks

(b) Even in this form, best hope fails for $n > 1$

due to singularities of $\mathcal{L}oc_n$.

Modification proposed by Arinkin - Gaitsgory

using Ind-Coherent Sheaves

(but this won't matter here).

The case $n = 1$:

autoduality of Jacobian

- $\mathrm{Bun}_1 = \mathrm{Pic}(C)$ Picard scheme $\supset \mathrm{Pic}^0(C) \cong A := \mathrm{Jac}(C)$

  abelian variety

- $\mathrm{Loc}_1 \longrightarrow \mathrm{Loc}_1 /\!\!/ \mathbb{G}_m = \mathrm{Loc}_1 =: A^\natural$

  moduli space of rk 1 local systems $(\mathcal{L}, \nabla)$ on A

Best hope boils down to Fourier-Mukai:

$$D^b_{coh}(\mathcal{D}_A) \xrightarrow{\ \sim\ } D^b_{coh}(\mathcal{O}_{A^\natural})$$

(Laumon - Rothstein)

This is a deformation of the "usual" FM:

<u>Def</u> (Deligne) A <u>z-connection</u> on a vbdle $\mathcal{E}$ is a map $\nabla : \mathcal{E} \to \Omega^1 \otimes \mathcal{E}$ sth $\nabla(f \cdot s) = z\, df \otimes s + f \nabla s$ $\forall f \in \mathcal{O}, s \in \mathcal{E}$.

Have

$$\begin{array}{l} \mathcal{E}(A) = \left\{ \begin{array}{l} \text{moduli space of} \\ \text{flat rk } 1\ \lambda\text{-conn. on } A \end{array} \right\} \\ \Big\downarrow z \\ \mathbb{A}^1 \end{array}$$

with $z^{-1}(1) = A^\natural$

$z^{-1}(0) = \mathrm{Higgs}_1 = \hat{A} \times B$

$(B = H^0(A, \Omega^1))$

Similarly, put $\mathcal{R}_A := \sum_{k \geq 0} F_k \mathcal{D}_A \cdot z^k \subseteq \mathcal{D}_A \otimes_{\mathbb{C}} \mathbb{C}[z]$

"Rees sheaf" on $A \times \mathbb{A}^1$

$$\Rightarrow \mathcal{R}_A|_{z=1} = \mathcal{D}_A$$

$$\mathcal{R}_A|_{z=0} = \mathrm{Sym}^\bullet(\mathcal{T}_A) \hat{=} \mathcal{O}_{A \times \underbrace{B}_{T^*A}}$$

We get

$$\begin{tikzcd}
D^b_{coh}(\mathcal{O}_{A^\natural}) \arrow[r, "\sim"] \arrow[d, rightsquigarrow] & D^b_{coh}(\mathcal{D}_A) \arrow[d, rightsquigarrow] \\
D^b_{coh}(\mathcal{O}_{\hat{A} \times B}) \arrow[r, "\sim", "\text{relative FM trafo}"'] & D^b_{coh}(\mathcal{O}_{A \times B})
\end{tikzcd}$$

for

$$\begin{tikzcd}
A \times B \arrow[dr] & & \hat{A} \times B \arrow[dl] \\
& B &
\end{tikzcd}$$

dual abelian schemes over B!

... What about $n > 1$?

$$\begin{tikzcd}
\mathcal{L}oc_n \arrow[d, rightsquigarrow, "z\text{-connections}"] \\
\mathcal{H}iggs^\circ_n
\end{tikzcd}$$

$\mathcal{H}iggs^\circ_n :=$ stack of flat rk n Higgs bdles

$$\begin{tikzcd}
\mathcal{D}_{Bun_n} \arrow[d, rightsquigarrow, "\text{Rees}"] \\
\mathrm{Sym}^\bullet(\mathcal{T}_{Bun_n}) \hat{=} \mathcal{O}_{T^*Bun_n}
\end{tikzcd}$$

Note: $T^*Bun_n \cong \mathrm{Higgs}_n$
$:= \mathcal{H}iggs_n /\!\!/ Z(G)$

Hitchin fibration: $\quad h : \mathrm{Higgs}_n \longrightarrow B := \bigoplus_{i=1}^{n} H^0(C, \omega_C^i)$

$(G = Gl_n)$

$$(\mathcal{E}, \Theta) \longmapsto \text{coeff's of char. polynomial of Higgs field } \Theta$$

$$b = (b_1, .., b_n) \in B \longleftrightarrow \left\{ \begin{array}{l} \tilde{C}_b := \{ z^n + b_1 z^{n-1} + .. + b_n = 0 \} \subset \mathrm{Tot}(\omega_C) \\ \downarrow \quad \text{"spectral cover"} \\ C \end{array} \right.$$

$\Delta := \{ b \in B \mid \tilde{C}_b \text{ singular} \}$ "discriminant" $\subset B$

$\Longrightarrow \quad h^{-1}(b) \simeq \mathrm{Pic}(\tilde{C}_b)$

In fact $\quad \mathcal{H}iggs_n^0 |_{B \setminus \Delta} \simeq (\mathrm{Higgs}_n |_{B \setminus \Delta})^D$

$$\mathcal{H}iggs_n |_{B \setminus \Delta} \simeq (\mathcal{H}iggs_n |_{B \setminus \Delta})^D$$

dual comm. gp stacks

$$\Longrightarrow \quad FM : \mathcal{D}^b_{coh}(\mathcal{O}_{\mathcal{H}iggs^0_n |_{B \setminus \Delta}}) \xrightarrow{\sim} \mathcal{D}^b_{coh}(\mathcal{O}_{\mathrm{Higgs}_n |_{B \setminus \Delta}})$$

$$\circlearrowleft \qquad\qquad \circlearrowleft$$

$$_{ab}W^i \qquad\qquad _{ab}H^i$$

"abelianized GLC"

Hope: Get back to GLC via nonabelian Hodge thy!

What about other groups?

G connected reductive gp $/\mathbb{C}$ $\updownarrow$ T max. torus	$\Longleftrightarrow$	root datum $(X, \Phi, X^\vee, \Phi^\vee)$

"Dynkin type + info on center"

where $X = \mathrm{Hom}(T, \mathbb{G}_m)$
$\cup$
Φ = roots in adjoint rep $G \circlearrowright \mathrm{Lie}(G)$

and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$
$\cup$
$\Phi^\vee$ = "coroots"
$= \{\alpha^\vee : \mathbb{G}_m \to T \mid$ restriction of $SL_2 \to G$ given by α, normalized s.th. $\langle \alpha, \alpha^\vee \rangle = 2 \}$

Def Langlands dual
${}^LG :=$ reductive gp with root datum $(X^\vee, \Phi^\vee, X, \Phi)$.

Ex.

G	GL_n	SL_n	SO_{2n+1}	SO_{2n}	simply conn.	...
${}^LG$	GL_n	PGL_n	Sp_{2n}	SO_{2n}	adjoint	...

Rem. $\pi_1({}^LG) = \mathrm{Hom}(Z(G), \mathbb{G}_m)$.

Replace:

- $Loc_n \rightsquigarrow Loc_G := \left\{ \begin{array}{l} \text{p'pal } G\text{-bdles } \mathcal{G} \to C \\ \text{with flat connection } \nabla \end{array} \right\}$

- $Bun_n \rightsquigarrow Bun_{{}^LG} := \{ {}^LG\text{-bdles on } C \}$

- Wilson operators:

  μ dominant wt of $G \longleftrightarrow$ irrep $V_\mu \in Rep(G)$

  $\longleftrightarrow$ associated bundle $V_\mu(\mathcal{G}) \ \forall \mathcal{G} \in Bun_G$

<u>Def</u> $W^\mu := V_\mu(\mathcal{G}^{univ}) \otimes (-) : D^b_{coh}(\mathcal{O}_{Loc_G}) \to D^b_{coh}(\mathcal{O}_{C \times Loc_G})$.

- Hecke operators:

$$\mathcal{H}ecke^\mu := \left\{ (\mathcal{G}, \mathcal{G}', \varphi, x) \;\middle|\; \begin{array}{l} \mathcal{G}, \mathcal{G}' \in Bun_{{}^LG},\ x \in C, \\ \varphi : \mathcal{G}|_{C \setminus x} \xrightarrow{\sim} \mathcal{G}'|_{C \setminus x} \text{ with poles} \\ \text{"bounded by } \mu\text{"} \end{array} \right\}$$

$$\begin{tikzcd} & \mathcal{H}ecke^\mu \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Bun_{{}^LG} & & C \times Bun_{{}^LG} \end{tikzcd}$$

ie $\forall \lambda$ dominant wt of ${}^LG$,

$V_\lambda(\mathcal{G}) \xrightarrow{\varphi} V_\lambda(\mathcal{G}') \otimes \mathcal{O}_C(\langle \mu, \lambda \rangle \cdot x)$

$\Longrightarrow$ Get $H^\mu : D^b_{coh}(\mathcal{D}_{Bun_{{}^LG}}) \longrightarrow D^b_{coh}(\mathcal{D}_{C \times Bun_{{}^LG}})$

$\quad\; \| $

$p_+(q^*(-) \otimes IC_{\mathcal{H}ecke^\mu})$

⚠ Usually $\mathcal{H}ecke^\mu$ singular for $G \neq Gl_n$
$\mu \neq \omega_i$ ...

Can again formulate Drinfeld's best hope. In particular:

$\underline{\text{GLC}}$: $\left\{ \begin{array}{l} \text{"generic"} \\ \mathcal{E} \in Loc_G(\mathbb{C}) \end{array} \right\} \xrightarrow{??} \left\{ \begin{array}{l} \text{Hecke eigen-}\mathcal{D}\text{-modules} \\ Aut_{\mathcal{E}} \text{ on } Bun_{{}^LG}, \text{ ie.} \\ H^\mu(Aut_{\mathcal{E}}) \simeq V_\mu(\mathcal{E}) \boxtimes Aut_{\mathcal{E}} \ \forall \mu \end{array} \right\}$

$$\Big\downarrow \text{NAHT} \qquad\qquad\qquad \Big\uparrow ??? \text{ NAHT}$$

$\underline{\text{abelianized}}$ $\underline{\text{GLC}}$: $\left\{ \begin{array}{l} \text{"generic"} \\ (\mathcal{E}, \theta) \in \mathcal{H}iggs_G(\mathbb{C}) \end{array} \right\} \xrightarrow[\text{FM}]{} \left\{ \begin{array}{l} \text{Hecke eigensheaf} \\ \text{on } \mathcal{H}iggs_{{}^LG} \end{array} \right\}$

For "aGLC" again consider Hitchin fibration:

$$h : \mathcal{H}iggs_G \longrightarrow \begin{array}{c} B \\ \cup \\ \Delta \end{array} \text{ discriminant}$$

$\underline{\text{Thm}}$ (Donagi-Gaitsgory) $\mathcal{H}iggs^\circ_G|_{B \setminus \Delta}$ is an abelian scheme over $B \setminus \Delta$.

$\underline{\text{Thm}}$ (Donagi-Pantev) $\exists$ iso $\begin{array}{ccc} B & \xrightarrow{\sim} & {}^LB \\ \cup & & \cup \\ \Delta & \xrightarrow{\sim} & {}^L\Delta \end{array}$

via which

$$\begin{array}{ccc} \mathcal{H}iggs_G & \overset{\sim}{\dashrightarrow} & (\mathcal{H}iggs_{{}^LG})^{\mathcal{D}} \\ \downarrow & & \downarrow \\ B \setminus \Delta & \xrightarrow{\sim} & {}^LB \setminus {}^L\Delta \end{array}$$

become dual commutative gp stacks.